International Journal of Theoretical Physics, Vol. 25, No. 3, 1986

Lie–Backlund Vector Fields for the Nonlinear System, $Q_t = AQ_{xx} + F(Q_x, Q)$

A. Roy Chowdhury[1] and Shibani Sen

Received October 16, 1985

We have analyzed the class of nonlinear second-order equations written as $Q_t = AQ_{xx} + F(Qx, Q)$ with $Q = \binom{u}{v}$ and A, F are, respectively, matrix and vector functions depending on Q, Q_x, from the point of view of Lie–Backlund vector fields. When the vector function F does not depend on Q_x, these equation set reduces to the coupled diffusion equations discussed by Steeb. But our generalized system encompasses a large class of physically meaning full nonlinear equations, such as (i) dispersive water waves and (ii) a completely anisotropic Heisenberg spin chain. We also exhibit a new nonlinear coupled system which do have nontrivial Lie–Backlund vector fields. Also our approach yields more information about the symmetry generators for a wider class of nonlinear equations than the function space approach of Fuchsteiner in a much simpler way.

1. INTRODUCTION

Classification of integrable equations from the point of view of symmetry structure forms one of the main techniques of analysis for the nonlinear equations (Fokas and Fuchsteiner, 1981). It has already been observed that nonlinear equations of second-order either forms the several type nonlinear Schrödinger equations or the celebrated diffusion equations (Steeb and Oevel, 1983). Here we have observed that it is possible to generalize these second-order nonlinear equations further, so that these include some more physically interesting cases, such as, anisotropic Heisenberg spin chain, dispersive water waves, etc. We have simultaneously determined the structure of the equations and also that of the Lie–Backlund vector fields. In this approach it becomes easier to fix up the form of the symmetry generators for complicated equations such as AHSC, which has been also tackled by Fuchsteiner (1979) by a function theoretic method.

[1]High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta—700 032, India.

0020-7748/86/0300-0319$05.00/0

2. FORMULATION

Let us denote by Q a two-component vector field, and let us consider a set of coupled equations:

$$Q_t = AQ_{xx} + F(Q_x, Q) \tag{1}$$

When the Q_x dependence of F is neglected we get the generalized class of diffusions equations, which has been treated by Steeb et al. In general A can also depend on (u, v) but here we consider only constant matrices

$$A = \begin{pmatrix} D_1, & D_1' \\ D_2, & D_2' \end{pmatrix}$$

so that the set (1) reads:

$$\begin{aligned} U_t &= D_1 D_2 + D_1' v_2 + f(u, v, u_1, v_1) \\ V_t &= D_2 V_2 + D_2' U_2 + g(u, v, u_1, v_1) \end{aligned} \tag{2}$$

In our ensuing computations we will have to consider also various differential consequences of (2). Let us assume that the Lie–Backlund vector field is written as

$$V = \eta_1(u, v)\frac{\partial}{\partial u} + \eta_2(u, v)\frac{\partial}{\partial v} \tag{3}$$

Consideration of invariance for the equations (2) leads to the following:

$$L_V F = 0, \qquad L_V G = 0 \tag{4}$$

where

$$\left.\begin{aligned} F &= U_t - D_1 u_2 - D_1' v_2 - f \\ G &= V_t - D_2 v_2 - D_2' u_2 - g \end{aligned}\right\} \tag{5}$$

L_V, denoting the Lie derivative (Ousjannikov, 1982) of F and G with respect to Lie–Backlund vector fields. But for the actual computation of (4) we require the extended form of the Vector fields, which reads

$$\begin{aligned} V_{ex} = {} & \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial v} + (D_x \eta_1)\frac{\partial}{\partial u_1} + (D_x \eta_2)\frac{\partial}{\partial v_1} \\ & + (D_x^2 \eta_1)\frac{\partial}{\partial u_2} + (D_x^2 \eta_2)\frac{\partial}{\partial v_2} \end{aligned} \tag{6}$$

where D_x stands for the total derivative (Anderson and Ibragimov, 1979)

$$D_x = \frac{\partial}{\partial x} + \sum_i u_{i+1}\frac{\partial}{\partial u_i} + \sum_i v_{i+1}\frac{\partial}{\partial v_i} \tag{7}$$

So that the equations determining the structure of η_1 and η_2 are

$$-\eta_1 f_u - \eta_2 f_v - (D_x\eta_1)f_{u_1} - (D_x\eta_2)f_{v_1} - D_1 D_x^2\eta_1 - D_1' D_x^2\eta_2 + D_t\eta_1 = 0 \tag{8}$$

$$-\eta_1 g_u - \eta_2 g_v - (D_x\eta_1)g_{u_1} - (D_x\eta_2)g_{v_1} - D_2' D_x^2\eta_1 - D_2(D_x^2\eta_2) + D_t\eta_2 = 0 \tag{9}$$

3. SOLUTION FOR (f, g) AND (η_1, η_2)

Since our equations are assumed to be linear in u_2 and v_2, we assume a linear structure for η_1 and η_2 in u_3, v_3 and write

$$\begin{aligned} \eta_1 &= a_1 u_3 + b_1 v_3 + h(u, u_1, u_2, v, v_1, v_2) \\ \eta_2 &= a_2 u_3 + b_2 v_3 + k(u, u_1, u_2, v, v_1, v_2) \end{aligned} \tag{10}$$

Conditions arising from the coefficients of u_4, v_4 are

$$\begin{aligned} f v_1 &= -g u_1, & \frac{\partial h}{\partial v_2} &= \frac{-\partial k}{\partial u_2} \\ f u_1 &= g v_1, & \frac{\partial h}{\partial u_2} &= \frac{\partial K}{\partial v_2} \end{aligned} \tag{11}$$

where we consider the situation

$$D_1 = D_2, \qquad a_1 = b_2, \qquad D_2' = -D_1', \qquad a_2 = -b_1$$

Equations (11) contains information for both the structure of the equation and the structure of the symmetry generator. For example we have

$$\begin{aligned} \eta_1 &= \alpha(u, v, u_1 v_1)u_2 + \beta(uu_1, vv_1)v_2 + a_1 u_3 - a_2 v_3 + h' \\ \eta_2 &= \alpha(u, v, u_1 v_1)v_2 - \beta(uu_1, vv_1)u_2 + a_2 u_3 + a_1 v_3 + k' \end{aligned} \tag{12}$$

Collecting coefficients of u_3, v_3, etc. yields

$$\begin{aligned} \alpha &= \phi(u, v)v_1 - \psi(u, v)u_1 \\ \beta &= \phi(u, v)u_1 + \psi(u, v)v_1 \end{aligned} \tag{13}$$

Substituting this information in the rest of the equations of (8) and (9) we get important conditions for the structure of equations

$$\begin{aligned} f_{u_1u_1u_1} &= 0 & f_{u_1v_1v_1} &= 0 \\ g_{u_1u_1u_1} &= 0 & g_{u_1v_1v_1} &= 0 \\ f_{u_1u_1v_1} &= 0 & f_{v_1v_1v_1} &= 0 \\ g_{u_1u_1v_1} &= 0 & g_{v_1v_1v_1} &= 0 \end{aligned} \tag{14}$$

So that we may put

$$f=\sigma(u,v)u_1^2+\xi(u,v)v_1^2+\theta(u,v)u_1v_1$$
$$g=\sigma'(u,v)u_1^2+\xi'(u,v)v_1^2+\theta'(u,v)u_1v_1 \tag{15}$$

Imparing (11) on (15) leads to

$$f=\sigma(u_1^2-v_1^2)-2\sigma' u_1v_1$$
$$g=\sigma'(u_1^2-v_1^2)+2\sigma u_1v_1 \tag{16}$$

Simultaneously we get equations of the following type for h' and K':

$$D_1\frac{\partial^2 h'}{\partial u_1^2}+D_1'\frac{\partial^2 k'}{\partial u_1^2}=\alpha f_{uu_1}+\beta g_{uu_1}+3a_1[f_{uu_1}+u_1f_{uu_1u_1}+\theta_1 f_{vu_1v_1}]$$
$$-3a_2[g_{uu_1}+u_1g_{uu_1u_1}+v_1g_{vu_1v_1}] \tag{17}$$

Suggesting that

$$h'=A(u,v)u_1^3+B(u,v)u_1^2v_1+C(u,v)u_1v_1^2+D(u,v)v_1^3$$
$$K'=A'(u,v)u_1^3+B'(u,v)u_1^2v_1+C'(u,v)u_1v_1^2+D'(u,v)v_1^3 \tag{18}$$

Next we put the complete structure of η^1, η^2 in the determining equations (8) and (9) and equating coefficients of u_1^2, v_1^2, u_1v_1; we obtain

$$2D_1'[3A-2B']=-2a_2[\sigma_u-\sigma_v']$$
$$2D_1'[C-3D']=-2a_2[-\sigma_u+\sigma_v'] \tag{19}$$
$$2D_1'[2B-2C']=-2a_2[-2\sigma_u'-2\sigma_v]$$

If we now set for simplicity's sake

$$D=D'=0$$

$$\eta_1=-\frac{a_2}{D_1'}(\sigma_u-\sigma_v')u_1^3+\frac{2a_2}{D_1'}(\sigma_v+\sigma_u')u_1^2v_1$$
$$+\frac{a_2}{D_1'}(\sigma_u-\sigma_v')u_1v_1^2+[\phi(u,v)v_1-\psi(u,v)u_1]u_2$$
$$+[\phi(u,v)u_1+\psi(u,v)v_1]v_2+a_1u_3-a_2V_3 \tag{20}$$

$$\eta_2=-\frac{a_2}{D_1'}(\sigma_v+\sigma_u')u_1^3-\frac{2a_2}{D_1'}(\sigma_u-\sigma_v')u_1^2v_1$$
$$+\frac{a_2}{D_1'}(\sigma_v+\sigma_u')u_1v_1^2+[\phi(u,v)u_1+\psi(u,v)v_1]u_2$$
$$+[\phi(u,v)v_1-\psi(u,v)u_1]v_2+a_2u_3+a_1V_3 \tag{21}$$

When

$$f = \sigma(u, v)(u_1^2 - V_1^2) - 2\sigma'(u, v)u_1 v_1$$
$$g = \sigma'(u, v)(u_1^2 + v_1^2) + 2\sigma(u, v)u_1 v_1 \tag{22}$$

So we observe that given any particular choice of σ, and σ', we can immediately write down the symmetry generators η_1 and η_2.

Example 1. Let us consider the system of equations

$$u_t = v_2 + \frac{2(u_1^2 - v_1^2)v - 4uu_1v_1}{1 + u^2 + v^2}$$
$$v_t = -u_2 + \frac{2(u_1^2 - v_1^2)u + 4vu_1v_1}{1 + u^2 + v^2} \tag{23}$$

for this system:

$$D_1 = 0, \qquad D_1' = 1, \qquad \sigma = \frac{2v}{1 + u^2 + v^2}, \qquad \sigma' = \frac{2u}{1 + u^2 + v^2}$$

so that it is easy to write down the form of η_1 and η_2. It is rather important to note this complicated-looking coupled system is not an artificial one, but a transformed form of a very important physical system, and that is the famous anisotropic Heisenberg spin chain, written as

$$S_t = S \times S_{xx} + S \times JS \tag{24}$$

where $S = \sigma_a S_a$, $J = \text{diag}(J_1, J_2, J_3)$ along with $S_1^2 + S_2^2 + S_3^2 = 1$. It is easily seen that by the mapping

$$S_1 + iS_2 = \frac{2Y}{1 + |Y|^2}, \qquad S_3 = \frac{1 - |Y|^2}{1 + |Y|^2} \tag{25}$$

it is converted to the form,

$$iY_t = Y_{xx} - \frac{2\bar{Y}Y_n^2}{1 + |Y|^2} + (J_2 - J_1)\left[\frac{Y^2}{4|Y|^2}\left(\frac{|Y|^2 - 1}{|Y|^2 + 1} - i\right)\right.$$
$$\left. + \frac{Y^{*2}}{4|Y|^2}\left(\frac{|Y|^2 - 1}{|Y|^2 + 1} + i\right) + \frac{1}{2}\frac{|Y|^2 - 1}{|Y|^2 + 1}\right]$$
$$+ (J_3 - J_1)\frac{|Y|^2 - 1}{|Y|^2 + 1}Y \tag{26}$$

If we now write $Y = U + iV$ then (26) is nothing but a coupled set like (8) and (9). So one can immediately write down the form of the generators from our formulas (21). Equation (23) is a special case of (26) when

$J_2 = J_1 = J_3$, so that it is a isotropic Heisenberg spin chain equation. In a recent publication. Fuchsteiner has elaborated a Lie-product-based method for the AHSC equation, but we feel that our approach, being less sophisticated and more straightforward, is easy to use yet yields the same information.

Example 2. As a second example we consider the system

$$u_t = u_2 - \frac{1}{2}\frac{u_1^2 + 2u_1 v_1}{u+v}$$
$$v_t = -v_2 + \frac{1}{2}\frac{v_1^2 + 2u_1 v_1}{u+v} \tag{27}$$

for this case $a_2 = b_1 = 0$ and $a_1 = b_2$. Similar considerations as before yield

$$\frac{\partial h}{\partial v_2} = \frac{\partial h}{\partial u_2} = 0; \qquad \frac{\partial h}{\partial u_2} = \frac{\partial k}{\partial v_2} \tag{28}$$

So that

$$\eta_1 = a_1 u_3 + \alpha(uvu_1v_1)u_2 + h(uvu_1v_1)$$
$$\eta_2 = a_1 v_3 + \alpha(uvu_1v_1)v_2 + k(uvu_1v_1) \tag{29}$$

Proceeding to the second step of the calculation

$$(D_1' = D_2' = 0,\ D_1 = -D_2 = 1)$$
$$2[\alpha_u u_1 + \alpha_v v_1] - 3a_1[u_1 f_{uu_1} + v_1 f_{vu_1}] = 0$$
$$2[\alpha_u u_1 + \alpha_v v_1] - 3a_1[u_1 g_{uu_1} + v_1 g_{vv_1}] = 0 \tag{30}$$
$$-2\frac{\partial h}{\partial v_1} = -3a_1[u_1 f_{uv_1} + v_1 f_{vv_1}]$$
$$2\frac{\partial k}{\partial u_1} = -3a_1[u_1 g_{uu_1} + v_1 g_{vu_1}]$$

which yields

$$f_{v_1v_1} = 0 = g_{u_1u_1}$$

So that

$$f = v_1 p + q, \qquad g = u_1 r + s$$

Finally we deduce $f_{u_1u_1u_1} = 0$ and $g_{v_1v_1v_1} = 0$, which immediately leads to

$$f = \tilde{p}(uu_1v)v_1 + \tilde{q}(u, v)u_1^2$$
$$g = \tilde{r}(u, v_1, v)u_1 + \tilde{s}(u, v)v_1^2 \tag{31}$$

with a proper choice of these functions we can generate now equation (27) and the corresponding symmetry generators are determined from (30).

Example 3. Lastly we consider a very important situation; given by the case of the dispersive water waves, written as (Kupershuidt, 1985)

$$u_t = -\frac{u_2}{2} + uu_1 + v_1$$
$$v_t = \frac{v^2}{2} - (u_1 v + uv_1) \tag{32}$$

In this situation we have

$$\eta_1 = a_1 u_3 + \alpha(u, v) u_2 + h(u_1 v_1 u_1 v_1)$$
$$\eta_2 = a_1 v_3 + \alpha(u, v) v_2 + K(u_1 v_1 4 u_1 v_1)$$
$$g_{u_1 u_1} = g_{u_1 v_1} = 0, \qquad f_{u_1 v_1} = f_{v_1 v_1} = 0$$

These conditions of f and g are easily satisfied by those in (32). Also we obtain

$$\frac{\partial h}{\partial v_1} = -3a_1[u_1 f_{uv_1} + v_1 f_{vv_1}]$$
$$\frac{\partial k}{\partial u_1} = 3a_1[u_1 g_{uu_1} + v_1 g_{uu_1}] \tag{33}$$

A calculation similar to our previous cases yields

$$f = \phi(u, v) u_1 + \psi(u, v) v_1$$
$$g = \phi'(u, v) u_1 - \phi(u, v) v_1$$

along with

$$\phi_u = -\psi'_u, \qquad \phi_v = -\psi'_v$$

So we can determine again (f, g) and also (h, k).

4. DISCUSSIONS

In the above we made a classification, on the basis of Lie–Backlund vector fields, for nonlinear equations of second order than the diffusion equations. The system of equations discussed are quite general to include various physical examples, in each case we have set up the requisite formulas for the symmetry generators.

ACKNOWLEDGMENT

One of the authors (S. Sen) is grateful to the Government of India (D.S.T) for a support through a Thrust Area Program.

REFERENCES

Fokas, A. and Fuchsteiner, B. (1981). *Physica*, **D4**, 47.

Steeb, W. H., and Oevel W. (1983). *Physica Scripta*, **28**, 513.

Fuchsteiner, B. (1979). *Nonlinear Analysis, Theory, Method, and Application*, **3**, 849.

Ovsjannikov, L. V. (1982). *Group Analysis of Differential Equations*, Academic Press, New York.

Anderson, R. L., and Ibragimov, N. Kh. (1979). *Lie Backlund Transformation in Application*, SIAM, Philadelphia.

Kupershuidt, B. A. (1985). *Communications in Mathematical Physics*, **99**, 51.